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The Painlevé test, hidden symmetries and the equation $y'' + yy' + ky^3 = 0$

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Abstract. For general values of the parameter, k, the equation $y'' + yy' + ky^3 = 0$ can be reduced to quadrature via a Lie algebraic approach, either direct or through hidden symmetries. For specific values of k, mostly in $(\frac{1}{9}, \frac{1}{8})$, the solution can be expressed in parametric form. For these values of k the equation passes the weak Painlevé test. For some other values of k the equation passes the Painlevé test, but the solution cannot in general be expressed parametrically.

1. Introduction

The differential equation

$$y'' + yy' + ky^3 = 0 \tag{1.1}$$

has occurred in studies in a variety of fields such as univalued functions defined by second order differential equations (Golubev 1950), the generalized Emden equation (Moreira 1984, Leach 1985), the Riccati equation (Chisholm and Common 1987) and in the modelling of the fusion of pellets (Erwin *et al* 1984), has itself been the object of study by Mahomed and Leach (1985), Leach *et al* (1988) and Abraham-Shrauner (1992) and as one of a class of equations by Bouquet *et al* (1991). These studies were mainly concerned with the Lie point symmetries of (1.1) and numerical properties of the solution for various values of the parameter k.

For general values of the parameter, k, equation (1.1) is invariant under the actions of the second extensions of the Lie point symmetries

$$G_1 = \frac{\partial}{\partial x} \tag{1.2}$$

associated with invariance under translation in the independent variable and

$$G_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \tag{1.3}$$

which is the generator of self-similar transformations. In the particular case that $k = \frac{1}{9}$ (1.1) possesses eight Lie point symmetries (Mahomed and Leach, 1985) with the algebra $sl(3, R)$ which implies that the equation is linearizable. The transformation is

$$Y = \frac{1}{2}x^2 - \frac{x}{y} \qquad X = x - \frac{1}{y}. \tag{1.4}$$

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Although the value of $k = \frac{1}{9}$ is significant in terms of the Lie algebraic properties of (1.1), this is not a critical value of k as far the behaviour of the solution is concerned. That value is $k = \frac{1}{8}$ (Leach *et al* 1988).

An interesting feature of (1.1) is that it has hidden symmetries associated with it no matter the value of k (Abraham-Shrauner 1992) in that it is related to a linear second-order equation by means of a non-local transformation.

The focal point of the study of any differential equation is the question of its integrability. Three approaches to this question are Lie analysis, Painlevé analysis and numerical studies. The purpose of this paper is to provide a unified treatment of (1.1) so that already known features are connected to some which have yet to be explored. In particular we compare two methods of reducing (1.1) to quadratures with the results of the Painlevé analysis of the equation. From this comparison we will see that the passing of the Painlevé or weak Painlevé test and the reduction to a parametric solution via the symmetries are closely related.

The parameter, k, which appears in (1.1) is written in that form for the convenience of the presentation of the equation. In the analysis which follows in the following sections it will be seen that another parameter, related to k, emerges naturally. As we wish to relate the reduction to quadrature to the results of the Painlevé analysis, we initiate that analysis first.

2. Painlevé analysis: initial considerations

Following the procedure outlined in Ramani *et al* (1989) we determine the leading-order behaviour by setting

$$y = \alpha\chi^{p} \tag{2.1}$$

where $\chi = x - x_0$, and find that there is a simple pole ($p - 1$), that all terms are dominant and that the coefficient α satisfies

$$k\alpha^2 - \alpha + 2 = 0. \tag{2.2}$$

The resonances are found by substituting

$$y = \alpha\chi^{-1} + \beta\chi^{r-1} \tag{2.3}$$

into the dominant terms (all in this case) and requiring that the coefficient of β be zero. When (2.2) is used, the resonances are found to be $r = -1$ (as required) and $r = 4 - \alpha$.

Equation (1.1) passes the Painlevé test if r is a positive integer which fixes α to one of the sequence of integers $3, 2, 1, -1, \ldots$. (The value $\alpha = 0$ is omitted due to (2.2).) One could contemplate the weak Painlevé test with α a rational number. However, for the moment we leave this analysis with two remarks. For $k > \frac{1}{8}$, α is complex. It has already been found (Leach *et al* 1988) that the critical value of k is $\frac{1}{8}$ and this is supported by the results so far of the Painlevé analysis. We have identified a parameter, α, related to k which plays a critical role in the analysis.

3. Reduction of order

Since (1.1) has two symmetries G_1, (1.2), and G_2, (1.3), with $[G_1, G_2] = G_1$, the usual reduction of order is through the normal subgroup, G_1, which yields an Abel equation of the second kind with the symmetry G_2 preserved in the new co-ordinates. This symmetry is used to transform the Abel equation into one of variables separable form. Under the transformation

$$u = \frac{1}{y} \qquad v = -\frac{y'}{y^2} \tag{3.1}$$

equation (1.1) becomes

$$uvv' = 2v^2 - v + k \tag{3.2}$$

which is immediately reduced to the quadrature

$$\int \frac{du}{u} = \frac{1}{2}\int \frac{v\,dv}{(v-\frac{1}{2})(v+(2-\alpha)/2\alpha)} \tag{3.3}$$

when (2.2) is taken into account. On integration (3.3) becomes

$$Ku^{2(4-\alpha)} = \left(v - \frac{1}{\alpha}\right)^{2-\alpha}\left(v + \frac{2-\alpha}{2\alpha}\right)^2 \tag{3.4}$$

where K is the constant of integration, except in the special cases $\alpha = 2(k=0)$ and $\alpha = 4(k=\frac{1}{8})$ which are

$$Ku^2 = v - \tfrac{1}{2} \tag{3.5}$$

and

$$Ku^2 = (v - \tfrac{1}{4})\exp[-\tfrac{1}{4}(v - \tfrac{1}{4})] \tag{3.6}$$

respectively. The reduction above is for $k \leq \frac{1}{8}$. For $k > \frac{1}{8}$ the quadrature of (3.2) gives

$$Ku^4 = (2v^2 - v + k)\exp\left\{\frac{1}{\sqrt{(8k-1)}}\tan^{-1}\left[\frac{2v-1}{\sqrt{(8k-1)}}\right]\right\}. \tag{3.7}$$

There still remains the further quadrature because of the transformation (3.1). Even if the immediate step of $v = u'$ is used, non-local inversion of (3.4) is not possible for general α.

4. Hidden symmetry reduction

The basic idea behind the hidden symmetry approach (Abraham-Shrauner 1993) is to *increase* the order of the given differential equation by one via a non-local transformation and then to reduce the order using one of the other symmetries to an equation of the same order as the original equation but with (one hopes) more symmetries. Equation (1.1) belongs to the class (Bouquet *et al* 1991)

$$y'' + y^m y' + ky^{2m+1} = 0 \tag{4.1}$$

for which the non-local transformation is a slight generalization of the Riccati transformation. It is

$$y = (1 + \frac{2}{m})^{1/m}(u'/u)^{1/m}. \tag{4.2}$$

In the case of (1.1) $m = 1$ and the resultant third-order equation is

$$u^2u''' + (9k-1)u'^3 = 0. \tag{4.3}$$

It is immediately obvious why (1.1) is easily integrated in the case of $k = \frac{1}{9}$.

Equation (4.3) has the three symmetries

$$G_1 = \frac{\partial}{\partial x} \tag{4.4}$$

$$G_2 = x\frac{\partial}{\partial x} \tag{4.5}$$

$$G_3 = u\frac{\partial}{\partial u} \tag{4.6}$$

with $[G_1, G_2] = G_1, [G_1, G_3] = 0$ and $[G_2, G_3] = 0$, i.e., the algebra is $A_1 \oplus A_2$ (compared with A_2 of (1.1)). The symmetry G_3 is a natural consequence of the Riccati transformation. The reduction of the self-similar symmetry (1.3) which contains both x and y terms to one only in x may have some connection with the fact that all terms in (1.1) are dominant in the Painlevé analysis.

When (4.4) is used to reduce the order of (4.3), the resulting equation is an Euler equation in the square of the new independent variable. A further transformation brings this into a linear constant coefficient form. Combining the two transformations we have

$$v = \log u \qquad w = u'^2 \tag{4.7}$$

and (4.3) becomes

$$w'' - w' + 2(9k - 1)w = 0\,. \tag{4.8}$$

Since (4.8) is a linear second-order equation, it is invariant under the actions of the generators of the eight-element algebra $sl(3, R)$.

The solution of (4.8) is

$$w = A\mathrm{e}^{\lambda_1 v} + B\mathrm{e}^{\lambda_2 v} \tag{4.9}$$

where

$$\lambda_{1,2}\tfrac{1}{2} \pm \delta \qquad \delta = \tfrac{3}{2}\sqrt{1 - 8k} \tag{4.10}$$

(except in the case $k = \frac{1}{8}$). The solution $u(x)$ of (4.3) is obtained from the inversion of the quadrature

$$x - x_0 = \int \frac{\mathrm{d}u}{u^{1/4}(Au^\delta + Bu^{-\delta})^{1/2}}\,. \tag{4.11}$$

A full discussion of the integration in (4.11) would necessitate the treatment of a number of cases resulting in a certain amount of repetition. To avoid this we consider only the case A and B both positive. The substitution

$$u^\delta = (B/A)^{1/2} \sinh \eta \tag{4.12}$$

brings (4.11) to the form

$$x - x_0 = \frac{1}{\delta A^{1/2}}\left(\frac{B}{A}\right)^{3/8\delta - 1/4} \int \sinh^{3/4\delta - 1/2} \eta \, \mathrm{d}\eta \tag{4.13}$$

which can be evaluated when the exponent on the sinh is an integer. The form of the integral varies with the oddness or evenness of the integer.

For an integer, $2n$, (4.10) gives

$$k = \frac{n(2n+1)}{(4n+1)^2} \tag{4.14}$$

and (Gradshteyn and Ryzhik (1980), equation 2.412.2)

$$x - x_0 = \frac{(4n+1)B^n}{3.2^{2n-1}A^{n+1/2}}\left\{(-1)^n \binom{2n}{n}\eta + 2\sum_{k=0}^{n-1}(-1)^k \binom{2n}{k}\frac{\sinh(2n-2k)\eta}{2n-2k}\right\}. \tag{4.15}$$

For an odd integer, $2n+1$, (4.10) gives

$$k = \frac{(n+1)(2n+1)}{(4n+3)^2} \tag{4.16}$$

and (Gradshteyn and Ryzhik (1980), equation 2.412.3)

$$x - x_0 = \frac{2(-1)^n(4n+3)B^{n+1/2}}{3A^n}\sum_{k=0}^{n}(-1)^k \binom{n}{k}\frac{\cosh^{2k+1}\eta}{2k+1}. \tag{4.17}$$

In terms of η

$$y = \frac{3A^{(n+1)/2}}{B^{n/2}}\frac{\cosh\eta}{\sinh^{n+1}\eta} \tag{4.18}$$

when

$$\delta = \frac{3}{2(2n+1)}. \tag{4.19}$$

In the case that n is an even integer, (4.18) with (4.13) provides a parametric solution and, when n is an odd integer, (4.18) with (4.17) does the same.

It is possible to eliminate η only in the cases $n = 0 (k = 0)$, $n = 1 (k = \frac{1}{9})$ and $n = 3 (k = 6/49)$. The even values of n suffer from the mixture of η and $\sinh\eta$ terms. Nevertheless the solution is well-defined for all integer n. We note that, except for the trivial case $k = 0$, the parameter k belongs to the open interval $(\frac{1}{9}, \frac{1}{8})$, i.e., between the value for which (1.1) is linearizable and the critical value.

If $k > \frac{1}{8}$, the solution of (4.8) is

$$w = \mathrm{e}^{v/2}(A\sin\delta v + B\cos\delta v) \tag{4.20}$$

where

$$\delta = \tfrac{3}{2}\sqrt{8k-1}. \tag{4.21}$$

The integral corresponding to (4.11) cannot be evaluated in terms of a finite combination of elementary functions.

5. Weak Painlevé property

In section 4 we saw that (1.1) could be solved in parametric form when

$$k = \frac{n(n+1)}{2(2n+1)^2} \qquad n = 0, 1, \ldots \tag{5.1}$$

for which, since $k < \frac{1}{8}$, the constant α in the Painlevé analysis takes the values

$$\alpha = \frac{2(2n+1)}{n+1}, \quad \frac{2(2n+1)}{n} \tag{5.2}$$

(with $n \neq 0$ in the latter case) and the Kowalevskaya exponents are

$$r = \frac{2}{n+1}, \quad -\frac{2}{n} \tag{5.3}$$

respectively.

For general n, r is non-integer and so (1.1) does not have the Painlevé property for these values of k. However, it is possible that (1.1) does have the weak Painlevé property since r is rational. In the case of the first resonance calculation shows that the expansion should be in powers $\chi^{2/(n+1)}$ rather that $\chi^{1/(n+1)}$. Without going into the details of what is a routine calculation we find that

$$y = \frac{2(2n+1)}{(n+1)}\frac{1}{\chi}\left\{1 + \left(\frac{\chi}{b}\right)^{2/(n+1)} + \frac{n^2+4n+1}{n+5}\left(\frac{\chi}{b}\right)^{4/(n+1)} + \frac{2n^4+19n^3+48n^2+23n+4}{2(n+5)(n+7)}\left(\frac{\chi}{b}\right)^{6/(n+1)} + \cdots\right\} \tag{5.4}$$

where b is the arbitrary constant which enters at the resonance. The weak Painlevé property is satisfied. As we have already seen in section 4, for these values of k the solution of (1.1) can be written in parametric form.

It has been claimed (Ablowitz *et al* 1980) that a negative resonance (apart from $r = -1$ which represents the arbitrariness of x_0 (Grammaticos *et al* 1982)) is purely formal. This is not always the case. When all terms in an equation are dominant, it is obvious that the leading term behaviour does not indicate whether the leading term is the lowest or highest power in the series. The existence of a negative resonance (apart from $r = -1$) indicates that we have the *highest* power and not the *lowest* power. Hence it is appropriate to insert the ansatz

$$y(\chi) = \chi^{-1}\sum_{k=0}^{\infty} a_k \chi^{-2k/n} \tag{5.5}$$

into (1.1). We emphasise that this is the correct interpretation of a negative resonance (apart from $r = -1$) only in the case that all terms are dominant. We find that

$$y = \frac{2(2n+1)}{n}\chi^{-1}\left\{1 + \left(\frac{b}{\chi}\right)^{2/n} + \frac{n^2-2n-2}{n-4}\left(\frac{b}{\chi}\right)^{4/n} + \frac{4n^4-17n^3-5n^2+24n+12}{2(n-4)(n-6)}\left(\frac{b}{\chi}\right)^{6/n} + \cdots\right\} \tag{5.6}$$

where again b is the arbitrary constant introduced at the resonance. As with all series solutions obtained using the Painlevé method, (5.6) is a formal solution. We conjecture that it represents an essential singularity at $\chi = 0$. This result, (5.6), is invalid for $n = 4, 6, \ldots$ at which values the arbitrary constant, b, must be zero. These values give particular solutions to (1.1). It is an easy calculation to show for n an even integer, $2m$, that

$$y = \frac{4m+1}{m\chi} \tag{5.7}$$

is a particular solution of (1.1).

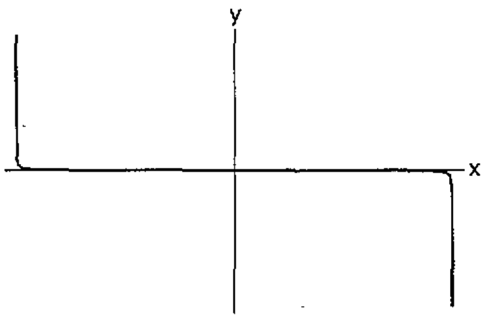

Figure 1. $k = \frac{1}{9}$; $y(0) = 0$, $y'(0) = -1$. Divergent solution.

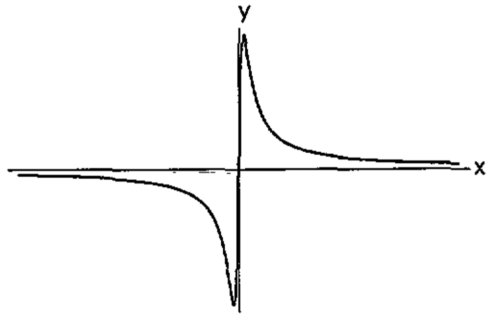

Figure 2. $k = \frac{1}{9}$; $y(0) = 0$, $y'(0) = 1$. Decaying solution.

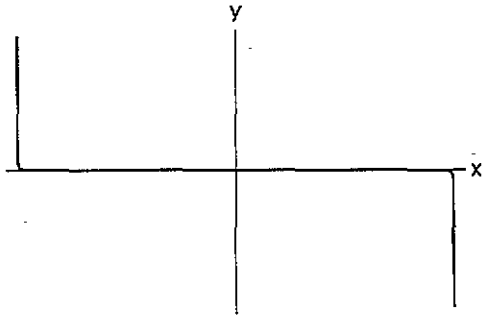

Figure 3. $k = \frac{1}{8}$; $y(0) = 0$, $y'(0) = -1$. Divergent solution indicates that $k = \frac{1}{8}$ is the limiting case of the $k < \frac{1}{8}$ behaviour.

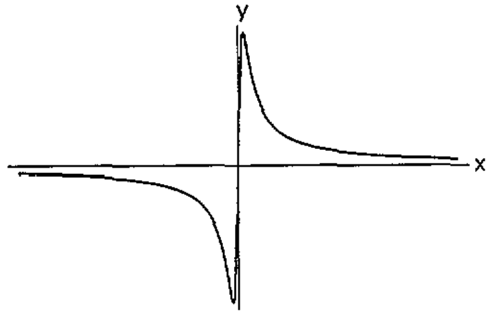

Figure 4. $k = \frac{1}{8}$; $y(0) = 0$, $y'(0) = 1$. Decaying solution indicates that $k = \frac{1}{8}$ is the limiting case of the $k < \frac{1}{8}$ behaviour.

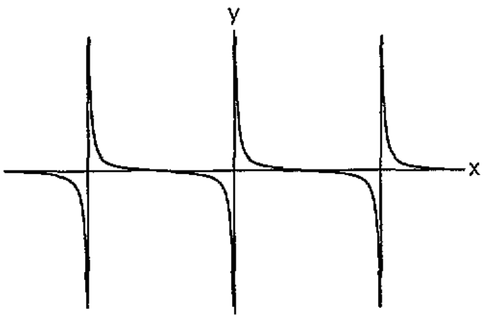

Figure 5. $k = \frac{1}{7}$; $y(0) = 0$, $y'(0) = 1$. The solution is oscillatory for all initial conditions.

6. Discussion

An interesting feature of (1.1) is the connection between the passing of the weak Painlevé test and the reduction of the solution to parametric form when k takes the values given in (5.1). Equation (1.1) can be reduced to quadratures via the symmetry approach—either direct or hidden—for any value of the parameter, k. However, the weak Painlevé property is found only for the particular values of k given by (5.1). We observe the curious property that, apart from the trivial case $k = 0$, this feature is restricted to values of k in $(\frac{1}{9}, \frac{1}{8})$. (Recall that, when $k = \frac{1}{9}$, the parameter is easily eliminated.) In terms of the behaviour of the solution, $k = \frac{1}{8}$ is the critical value in that the solution passes from non-oscillatory to oscillatory (see figures 1 to 5). In Leach *et al* (1988) it was stressed that it was this value rather than $k = \frac{1}{9}$ which was the critical value. However, the present analysis strongly suggests that $k = \frac{1}{9}$ is also a critical value although in a sense different to normal usage.

In the initial treatment of the Painlevé analysis we noted that the Kowalevskaya exponent was $4 - \alpha$. Clearly the Painlevé test is satisfied for $\alpha = 3, 2, 1, -1, \ldots$ (α cannot be zero). By way of example in the particular case $\alpha = 1$ we find that

$$y = \frac{1}{\chi}\left[1 + (b\chi)^3 + \frac{4}{27}(b\chi)^6 + \frac{53}{27}(b\chi)^9 + \cdots\right] \tag{6.1}$$

i.e. y is essentially a function in the variable $[b(x - x_0)]^3$. It appears that for general integer α, the expansion is in powers of $[b(x - x_0)]^{4-\alpha}$.

If one performs the reduction of order via the hidden symmetry method with the substitution $y = 4u'/u$, the resulting quadrature is

$$x - x_0 = \int \frac{\mathrm{d}u}{(Au^q + Bu^{-q})^{1/2}} \tag{6.2}$$

where $q = 2(4 - \alpha)/\alpha$. This can be evaluated in closed form for $\alpha = 3, 2, 1$ which correspond to $k = \frac{1}{9}, 0$ and -1 respectively. In contrast to the situation when the weak Painlevé test is satisfied, the passing of the Painlevé test does not coincide with expression of the solution in parametric form (except for the special cases noted above).

Acknowledgments

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References

Ablowitz M J, Ramani A and Segur H 1980 *J. Math. Phys.* **21** 715
Abraham-Shrauner B 1992 *J. Math. Phys.* submitted
Bouquet S E, Feix M R and Leach P G L 1991 *J. Math. Phys.* **32** 1480
Chisholm J S R and Common A K 1987 *J. Phys. A: Math. Gen.* **20** 5459
Erwin V J, Ames W F and Adams E 1984 *Wave Phenomena: Modern Theory and Applications* ed C Rogers and J B Moodie (Amsterdam: North-Holland)
Gobulev V V 1950 *Lectures on Analytical Theory of Differential Equations* (Moscow: Gostekhizdat)
Gradshteyn I S and Ryzhik I M 1981 *Table of Integrals, Series, and Products* (New York: Academic)
Grammaticos B, Dorizzi B and Padjen R 1982 *Phys. Lett.* **89A** 111
Leach P G L 1985 *J. Math. Phys.* **26** 2510
Leach P G L, Feix M R and Bouquet S E 1988 *J. Math. Phys.* **29** 2563
Mahomed F M and Leach P G L 1985 *Quaest. Math.* **8** 241
Moreira I C 1984 *Hadron. J.* **7** 475
Ramani A, Grammaticos B and Bountis T 1989 *Phys. Rep.* **180** 159